**General Relativity Geometry**

**Einstein’s Field Equations**

Guess should refer to that Tensor file…Following up on the insight suggested by the first postulate, we want to determine the curvature of space-time, and ultimately, the metric. So we’ll follow Einstein and make a heuristic inference as to what exactly determines the curvature. First, the curvature of space-time is described by the Riemann curvature tensor, Rαβδγ, and by its various contractions: the Ricci tensor Rαβ = Rδαβδ and the Ricci scalar R = Rαα. Seeking the simplest possible equations of gravity, we would first consider the possibility of having a scalar equation of gravity:



Now Newton’s theory of gravity states that



and so we might suspect that it is the mass density that determines the curvature. But knowing that mass is convertible into energy and vice versa we should probably say rather that it is the rest energy density that determines the curvature. So we might postulate:



where ρ0 is the mass density causing the gravitational field. But such an equation is a little wanting – for instance it places the rest frame of the mass as the preferred frame and we should have an equation which doesn’t give preference to any coordinate system. And in any case, such an equation was tested initially but was found to make false predictions – namely it didn’t correctly predict the advance of the perihelion of Mercury; instead it predicted it should retrograde. So going to the next higher up equation we could have:



The right hand side must be a tensor, and a natural candidate which includes the object’s mass, energy, etc., is the stress-energy tensor Tαβ. So we might postulate:



where,



But this can’t be quite right because we know that the stress-energy tensor satisfies the equation:



while Rαβ does not, generally. But this isn’t the end – we can perhaps add to the LHS other terms which would make the space-time divergence equal to 0. So we’ll add to the LHS other terms which are related to the curvature:



Now let’s see if we can choose the coefficients to make the LHS satisfy the divergence identity. We want:



First let’s show that the 4-divergence of **g** is zero regardless.



And so now we have:



And now let’s work on these last terms. To help us out, we’ll use the Bianchi identity:



Applying the metric to both sides, and using the fact that **g**’s derivative is zero we can write:



This is called the first contracted Bianchi identity. Now let’s contract again over the β and ν indices.



This is called the twice-contracted Bianchi identity by the way. So filling this into our equation above we get:



So now that we have the required μ, we have an acceptable set of equations called the Einstein equation:



The constant Λ is called the cosmological constant, which Einstein originally discarded for aesthetic reasons, put back in to produce a static universe, and then took out again after it was discovered the universe is actually expanding and not static. It has been recently put back in *again* because it has been found the universe’s rate of expansion is actually *accelerating*. Λ is considered to be the source term for dark energy, which has/had been thought to be due to quantum field vacuum fluctuations. In that spirit we’d write this as:



But estimates of Λ based on this idea don’t match up at all (they are 100 orders of magnitude too big) with experimental measurements.

The constant k would be so far undetermined. But it can be inferred by calculating the field equations for weak gravitational fields (the Newtonian limit) and choosing k to match Newton’s gravitational field equations. We’ll see how this is done in a bit. Another point, the stress-energy tensor has been previously discussed for matter, and is given by (recall ρ0 is rest density, and ε0 = ρ0c2 is rest energy density):



for dust, and



for ‘fluids’ [note we updated → ]. But you will know upon a study of EM, fields also have energy and momentum, and therefore a stress-energy tensor of their own, which is defined in the same way. And so we should include this in **T** as well. So for instance, the stress-energy tensor for EM fields is (see EM folder/Symmetries and Conservation Laws):



For future reference, we can put the Ricci scalar in terms of the trace of the stress-energy tensor as follows.



So we pause to note this interesting fact, that the curvature of space-time is governed by the trace of the energy tensor, and the cosmological constant.



and this allows us to write Einstein’s equation as:



so,



**Interpretation of the Metric**

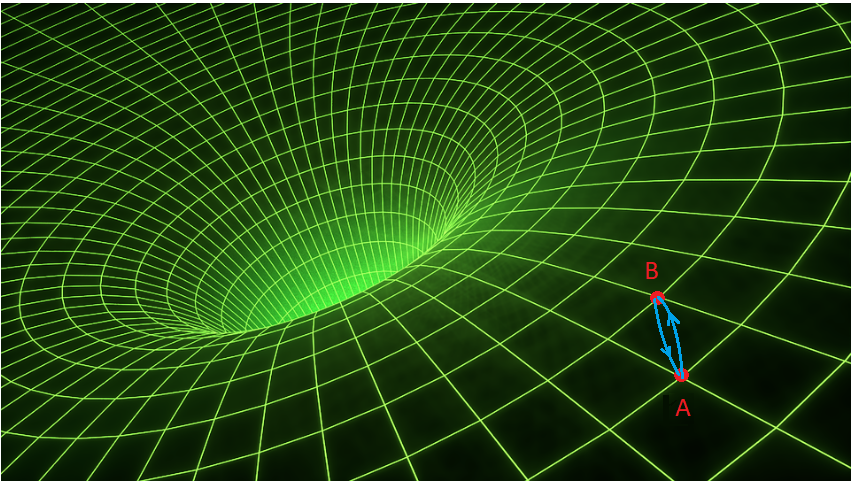
First I’ll note that for a given matter-energy distribution we should have many different equivalent solutions to Einstein’s equations (i.e. can express the metric in many different coordinate systems). For instance any freely falling observer should be able to write a metric solution to the equations, which according to him, is locally flat and comoving with him at instantaneous velocity **v**. For instance, if there is no matter-energy present (T = 0), then g = η would be a solution (if Λ = 0). And any Lorentz coordinate transformation [Lorentz boost to a new velocity **v**], which would correspond to an observer ‘free-falling’ at constant velocity **v**, would have the same metric. And further, we could slice up the spatial part of the manifold any way we want – Cartesian, spherical, etc. If we have matter-energy present, then we can get a metric solution by whatever means. But then we can, somehow, change coordinates to one which is locally flat and moving with velocity **v**, at any other given point in space-time [the latter part can at least be done approximately with a local Lorentz boost]. And like in SR, we can of course slice up the spatial manifold anyway we want.

Now I want to consider how to interpret a metric, once it has been solved for via Einstein’s equation. Specifically want to figure out how to know what the physical time and distance between two points in the manifold are. Remember that this was something we had to work out even in Special Relativity, as the distance in space-time ds2 = dt2 – dx2 was a combination of physical time-distance and physical spatial distance. So likewise we have to figure out how to extract these things. So consider a general metric,



where g is a symmetric matrix (it must be), and dx0 = cdt of course. The coefficients gαβ can be interpreted as α·β, where these are the covariant basis vectors in those respective directions.

These xα are called book-keeping coordinates, to distinguish them from any actual physical coordinate. These are just coordinates on a manifold, like the radial coordinates were on the 2-sphere in the tensor notes. And their intervals dx0, or dxidxi do not necessarily equate to any physical time or space interval. Still, at any given point in our manifold, a book-keeping coordinate displacement is commensurate to some physical time and space displacement, which we’ll call d and d. And we’d like to know what it is.



First, a preliminary result. Consider a point in our manifold, and an event that begins and ends at that same spatial point, like a light ray emitted from A towards a point differentially far away, B, and reflected back. Our metric would describe the invariant interval associated with this event as:



Now remember that our space is everywhere locally flat, and so can be locally described by a Minkowski metric. So transform to such a set of coordinates. In this reference frame, locally at rest, the time it takes for this even to transpire would be described as:



Equating the two, we have an equation that relates proper local time to book-keeping time,



Now let’s consider physical distance. We know, by going back to our local Minkowski metric, that the light ray would be locally described as traveling at the speed of light, and so the distance between AB can be considered to just be d = cdτ/2, where dτ is the time it takes for the signal to emanate from and return to point A, according to the local frame. Now in our local Minkowski frame, recall that light will satisfy the equation ds2 = 0 (by virtue of the fact that it travels at the speed of light), and so this will also be true when transforming back to our book-keeping coordinates. Going back to our original frame then, we’d have, for light’s outward journey,



and for the backward journey, setting d**x**BA = -d**x**AB, we have:



adding the two times together we get:



So the distance between these two points is:



Dropping the subscripts, we can say that the physical space metric is given by:



Now let’s consider physical time, which would give us the time it takes for the light ray to travel from A to B. We get this by comparing the invariant element between A and B (leaving off the subscripts) as expressed in book-keeping coordinates and physical coordinates:



and so we have:



and finally,



And as explicitly invoked above, we can now write the metric as:



For what it’s worth, one can now define ‘physical’ coordinates at any book-keeping coordinate, which diagonalize the metric.



where specifically,

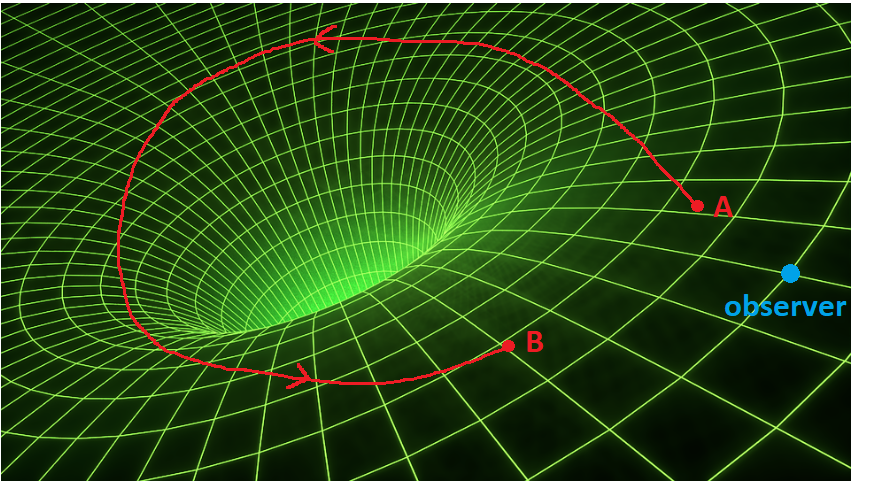


This is just a local transformation though. If it turns out that d0 is an exact differential, then that means we can write the physical time as a function of book-keeping time and space, = (t,**x**), which means there is a coordinate transformation, *t* = *t*(,**x**) to a frame in which we can define a global time coordinate, whereby clocks at every point progress at the same rate – no time dilation. Furthermore, in terms of the variables 0, **x**i we can then write the metric as:



One can apparently do this for a special few metrics, like the Robertson-Walker metric which defines the structure of space-time on a universal scale. But we cannot do this for cases like a Schwarzild metric, which describes the space in and around a star.

A couple questions now… Consider some path through space xα(s) that begins and ends at xAα and xBα respectively.



What is the physical length of this path? This would seem to be:



where gαβ(s) is parameterized by s through g’s dependence on the book-keeping coordinates xα(s). Now how long would this path take to be traversed, according to the path-taker? This would be:



Another question, ‘how long would this appear to take to be traversed to an observer elsewhere’? In this case, I think we would use the *observer’s* path through space-time xα(s), in place of the object’s path, and integrate *their* physical time between the same two limits.



Typically, the observer wouldn’t be going anywhere, and so this would simplify to:



And if it were the case that g00 is time-independent, like with the Schwarzild metric, then this would just be:



g00 *would* typically depend on the (observer’s) position.