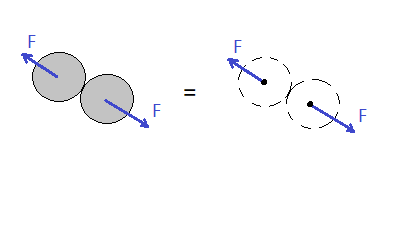
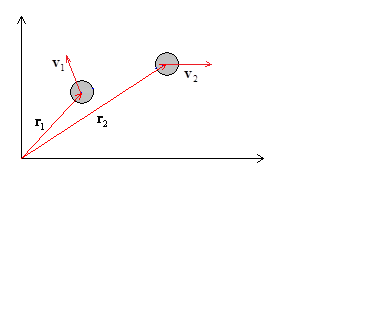
**2 Particle Hard Shell Interaction**

Now let’s consider a ‘simpler’ case, that of a billiard ball like interaction where particles experience a super hard repulsive barrier that they can’t penetrate. The force acts at the edge,



and if we treat the particles as effectively point-particles, rather than solid spheres, then we can say the force acts at a distance d = 2R away from the particle itself. Now we’d like to know something of the trajectory of these particles when they collide. All of the analysis we did for the gravity case holds here too, up to the part where we plug in the actual form of the force.

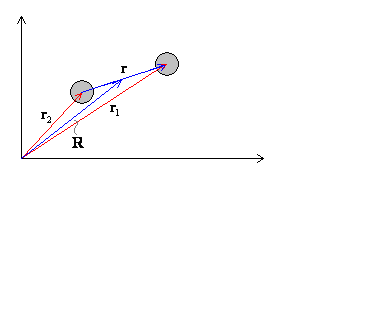


(obviously not going to do anything in this picture, but if they were different trajectories, they could collide). So we start with Lagrangian,



Then we switch to center of mass coordinates:





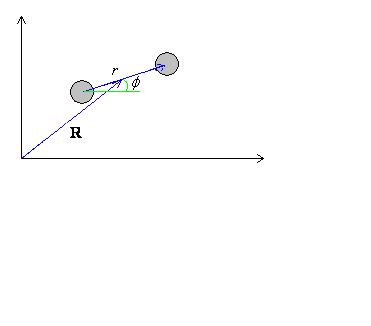
And get, ultimately:



As with last time, we observe that since U(r) is a function of r alone (not any angles), L possesses rotational symmetry in the coordinate **r**, which means that the angular momentum **L** of the mass, μ, is conserved (recall Noether’s theorem). This means that



is constant, which implies that **r** is confined to a plane – think about it. Therefore the relative coordinate **r** has only two degrees of freedom – its magnitude and the angle it makes in its plane of rotation.



Therefore we can write L as:



Now we’re ready to get the equation of motion of **R**, and **r**. As before, for **R** we get:



Solving for **R** we get:



So this says that the motion of the center of mass is free (it doesn’t accelerate). And then for r, φ, we have:



This quantity that is constant is the relative angular momentum of the particles which we call ℓ. Then we have:



Now apropos r,



which we recognize as N2L (in the radial direction):



And we can translate this into an energy conservation equation like before, to get:



where,



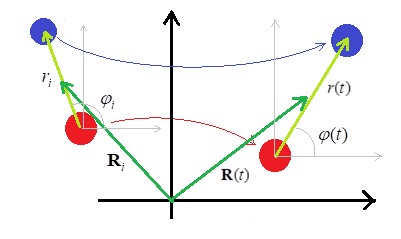
and we observe that there is no potential energy of interaction initially, since the balls will not be in contact initially. So now we have our equations to solve, once we fill in U(r). But I think this is not the best way to go since it’s hard to write a U(r) for this force. Could write F(r) = γδ(r – d) perhaps, where γ → ∞ ultimately, and δ is delta function. Then U(r) = -γθ(r < d) where θ is Heaviside step function. But this seems like too much trouble since we’d need to keep γ finite, solve the problem, and then take the limit γ → ∞. So going to back up to:



The dynamics encoded in Lrel is just that of a particle of mass μ traveling in the force field represented by U(r), centered at the origin. And we should be able to work out the dynamics of such a particle from other considerations. I think we just get ‘reflection’….

**Collisions**

So let’s work out the following scenario:



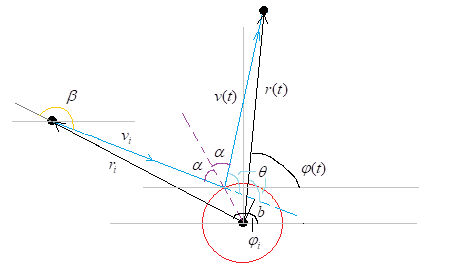
We have particles with initial positions and velocities, and, supposing they collide, we’d like to figure out their final positions and velocities as a function of time. So the first thing we do is go to center of mass / relative coordinates as above:



And the equations for **r**(t) and **R**(t) are:



and **i**, **j**, are unit vectors in the plane of the motion, as illustrated above. Given an initial ri, φi, I believe the trajectory is just as illustrated below – the reduced mass particle just ‘reflects’ off the target. Note the red circle has diameter d = 2R where R is the radius of the particles.



So particle starts off at **r**i = (ri, φi) and **v**i. β is the angle between the initial position and initial velocity vectors. It will be greater than 90o presumably, or else it’d be heading away from the target. So then it reflects off of the other guy following trajectory **r**(t) = (r(t),φ(t)) and velocity **v**(t). Magnitude of **v**(t) is same as initial, vi. Let’s specialize to finding the asymptotic limit of **v**(t). In the asymtotic limit, the two angles marked θ will be the same, as the blue and black lines become almost parallel. And then we can say, vis a vis the triangle made by ri, r(t) and vi, that: θ + (π – β) + φi – φf = π → φf = φi + θ – β. Now looking at the blue line defining the impact parameter, we have:



where b is the ‘impact parameter’ or point of closest approach, given by b = risinβ and β = cos-1(i·i). Well really, since the incoming guy will miss the target if b > d, we have:



So we can say:



And then let’s work out the final velocities of the particles. So we have:



And so the final velocities will be:



Since vi = |**v**1i – **v**2i|, and defining θiv = φi­ – β as the initial orientation of the relative velocity vector, we can write this as:



And of course this is just the form we said we should get based on our generic conservation laws analysis. Equation simplifies a lot for head on collisions. Then β = π, θ = π. And basically the ball just rebounds along opposite relative direction. And we’d have:



If equal masses, then:



In 1D, the generic case reduces to:

