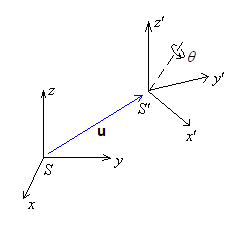
**Lorentz Transformation 2**

**Mathematical Formulation of Transformation**

So far we’ve only considered Lorentz transformations in one direction – along the x-axis. But suppose we have more complicated situations. For instance, suppose that S′ is rotated w/r to S by some angle **θ** and is also moving off with some velocity **u**, all such that the origins still coincide at t = t´ = 0. Then what is the relationship between measurements in S′ and S?



It wouldn’t be hard to work out the transformation for the velocity part, but the transformation for the rotation part too might be a little difficult. To proceed, we’ll actually use another procedure quite different from the one employed via the time-dilation and length – contraction formulas. This is the more modern method. So generally speaking we seek a transformation of coordinates from xα to xα′ via:



or more simply,



where ,  are column vectors and **Λ** is the Lorentz transformation matrix. This transformation matrix should be 4 dimensional, and should also conserve the magnitude of the space-time vector  [an equivalent condition probably, is that it preserve the metric]. We will seek the most general matrix which has these properties. To that end, we’ll find that it is easiest to determine the properties of the infinitesimal transformation matrix. That is, we expand the transformation matrix in a power series in ‘ε’. The full transformation matrix corresponds to ε = 1.



Now let’s consider an infinitesimal transformation out to first order in ε.



and we demand that it preserve the magnitude of the space-time vector:



And so we must have that:



Now since this equation must hold for all possible  it must be the case that:



Let’s work out this equation and see what it entails.



This implies that each diagonal element is 0. And furthermore, it implies that the lower 3×3 block part of the matrix is anti-symmetric, while the first row and column are symmetric. So incorporating this symmetry we can write that **L** is most generally:



So there are a total of 6 undetermined elements of **L** and we cannot be any more specific per se′ than this. And should not surprise us because **L** must contain information about **u** and **ω** which is a total of 6 free parameters. OK, at this point it is customary to express the matrix **L** in terms its 6 degrees of freedom thusly:



where the K and S matrices are:



Let’s consider some of the properties of these matrices. First,



So they are basically ones on the diagonal. And secondly we have:



From these relationships all higher powers follow. We can also easily, if laboriously, demonstrate the following commutation relations



From these commutation relationships, we should see that the S matrices are probably rotation matrices, while the K’s are probably boost matrices. This suspicion will be born out when we calculate the finite transition matrix, **Λ**. So to that end…consider the following. Out to first order we have:



We don’t actually need to know what **M** and **N**, etc., are. **L** suffices through the following argument. We cut the full transformation into N small pieces. The transformation matrix for each small piece is:



and so the full transformation matrix is N repeated small transformation matrices:



What is this limit? Well take the ln of both sides:



exponentiating both sides, and filling in what **L** is we have our desired result:



This clearly doesn’t look like our previous result for **Λ**. But we’ll do a special case of a pure boost to see that it does indeed reduce to our earlier expression. This will help us identify what the parameters L0i should be. And then we’ll do a pure rotation to help us identify what the parameters Lij should be.

**Special case: pure boost in x direction**

If there is no rotation then **ω** = 0, and moreover if there is only a boost in the x-direction then we have:



Taylor expanding…



And writing this out explicitly in terms of the K1 matrix we have:



Comparing to the already known Lorentz transformation in the x-direction we have:



and so we see that L01 is simply the ‘rapidity’, defined in a previous file,



Doing similar calculations in the y and z directions we would find that:



**Special case: pure rotation about the z-axis**

Now let’s calculate **Λ** for a pure rotation about the z-axis. Then we have:



Taylor expanding…



Filling in the matrices we have:



Making note that the matrix inducing rotations about the z-axis is, in regular 3D:



and that

it seems apparent that we should interpret L23 as θ3, the angle that S′ is rotated about the z-axis. Doing similar calculations in the other directions we will see that generally,



**General boost to speed u, and rotation θ**

Now we’re in position to determine the general form of the Lorentz transformation matrix. filling in our results we have:



Defining the vectors.



we can write this more concisely as:



Explicitly figuring out what Λ is can require some effort however. I’ll leave it to you to determine that:



where as usual: ξ = tanh-1β and i = βi/β. In terms of the more conventional variables, this comes to:



as for what Λ(**β**,**θ**) looks like, well, good luck on that.