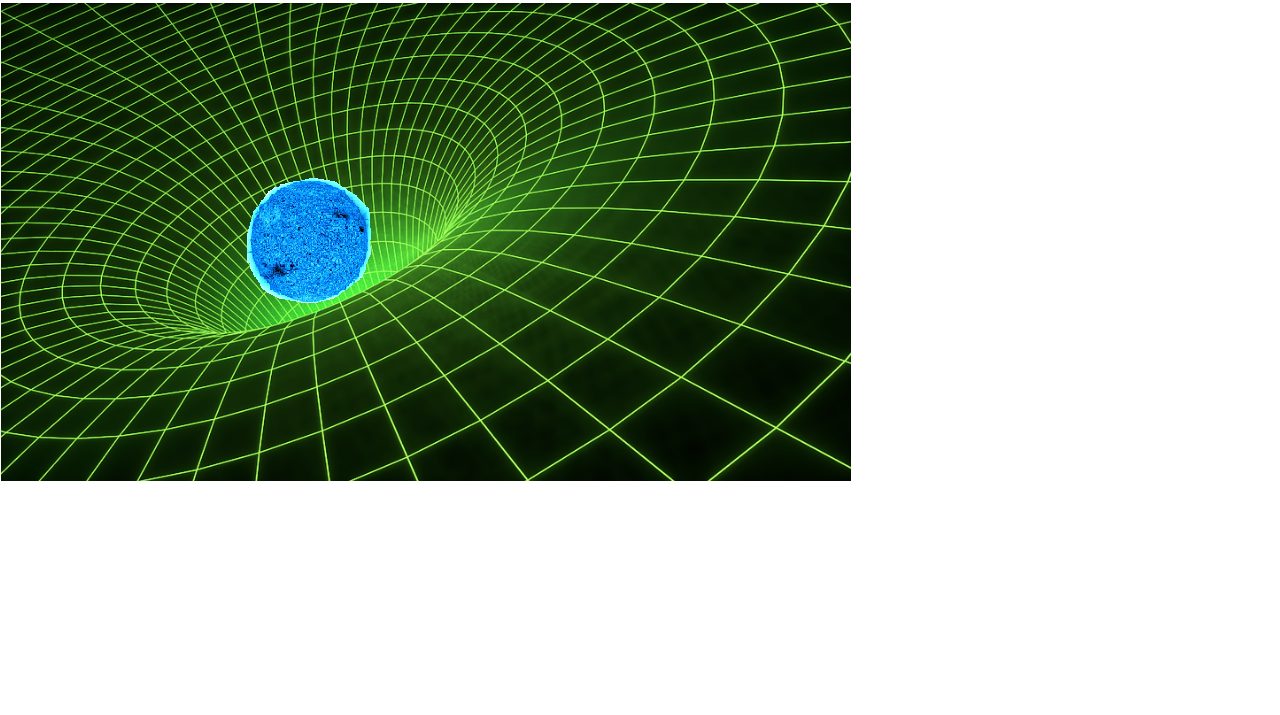
**Stars**

Now let’s explore another special case – stars. More generally, we’ll simply look at spherically symmetric arrangements, but these always describe stars.



Again we’ll start with Einstein’s equations:



Since stars are spherically symmetric, we will assume spherical symmetry in our metric ansatz. Additionally, we’ll assume that in our reference frame, the metric is independent of time, and also possesses time-reversal symmetry so that the star basically isn’t rotating. Let 0 go in the time-like direction, r go in the radial direction, θ in the theta directions, and φ in the phi direction. In that case we can write our metric as:



Note that t, r, θ, φ are coordinates in the time-like and radial-like, and theta-like, phi-like directions. But the former two are not identical to physical time and physical radial distance of course. What is the LHS of the Einstein equation? Let’s work it out. Our metric tensor is:



where T = T(r), R = R(r), Θ = r2, Φ = r2sin2θ, and e00, e11, e22, e33 are just place holders indicating the position of the elements in the matrix. Proceeding,



and



and so we should have for Γ,



The Ricci tensor is:



Now taking the derivative and trace at the same time, to form the first term in the Ricci tensor we have:



Next,



Next,



Next, and finally,



And now adding these together…we have:



Grouping together,



In particular, we’ve got:



The next term is:



and the next,



and the R33 guy,



The off-diagonal elements are zero, as can see…



and,



So so far, our metric is:



and our curvature tensor is so far:



Now we must fill these expressions into the Einstein equation.



**Exterior solution**

Now, if we’re looking for solutions outside the star, then Tμν = 0. And so our equations reduce to simply,



Let’s try to solve these equations (the last equation is dependent on the others and doesn’t yield a new condition).



The first thing we might try to do is eliminate the Tr2 terms from the top two equations. So let’s multiply the top by R/T and subtract it from the bottom. This will give us:



These equations imply that TR = constant. Let’s denote this constant by α. Then we have T = α/R. Now substituting this equation into the equation for R22 we get:



where k is some constant to be determined. It follows therefore that:



Now we must work out what k and α are supposed to be. Let’s compare to the weak field limit of Einstein’s equations:



and of course, using the φ for a star, far from a star we should get that



This suggests that we have α = – c2, and k = 2GM/c2. If so then we have the following metric outside a star,



**Interpretation of the time coordinate and radial coordinate**

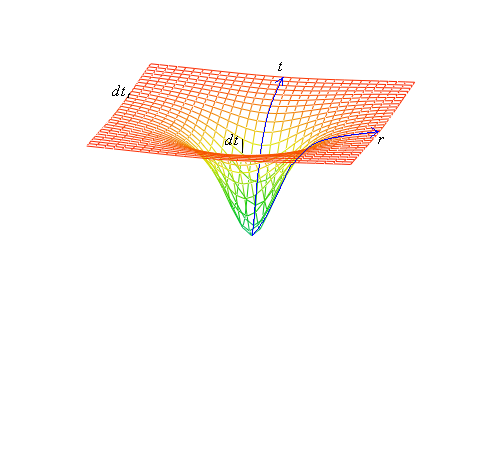
So the time-like coordinate does not represent actual time, and neither does the radial coordinate represent radial distance. Consider a stationary observer out at infinity. We’ll note that as r → ∞, g00 → -1. This tells us that physical time between events A and B for an observer relates to the coordinate time between those events according to [where the implicit xα(s) is the path of our stationary observer] :



and so basically, the physical time for an infinitely far removed person is the same as book-keeping time. We’ll also observe time dilation. Consider an event that begins and ends at book-keeping time tA,B respectively. What is the physical time interval as measured by someone at radial position r? This would be:



So the same event will seem to take less and less time, the further you go into the well. We can think of this as because the time coordinate is getting stretched out as we go into the well, kind of illustrated below:



We’ll note that as the observer gets close to r = 2μ, events which take no time at all for him, will take close to ∞ time for infinitely far removed observers. This is the event horizon, and the radius is called the Schwarzild radius.



Of course for this formula to apply, this radius must be outside the star. So we could say that if a star’s radius were less than this value, then it would possess a so-called event horizon. The required radius for the Earth, for example, would be about 2mm. Speaking of radii, we should note that this isn’t the actual ‘radius’ as in distance from center of star to event horizon. Radial distances would be calculated via:



Now we don’t actually know the metric inside the star, so we can’t exactly calculate *this*. But outside the star we’d have:



This has some result I won’t bother with.

**Red shift**

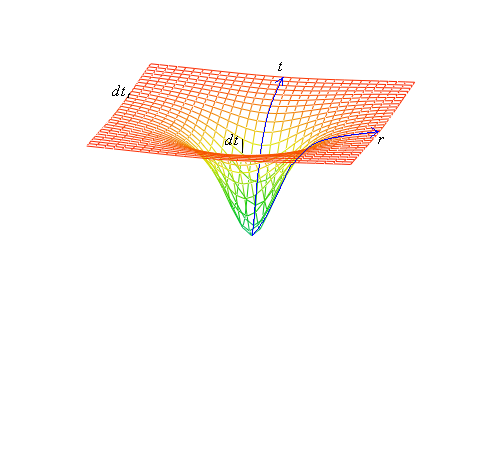
Let’s calculate the shift in wavelength of a photon traveling up the gravitational field. To make progress on this front, we’ll note that since the metric doesn’t depend on time, p0 is a conserved quantity. This is:



So the energy of the photon is not constant (nor would it be for a particle traveling up a gravitational field). The basic reason for this is that as the photon travels up the field, its wavelength stretches out as space stretches out further away from the star. Filling in E = hν, and taking the ratio of energies at different points we get:



As you can see, the further away from the star the particle gets, the lower the frequency (longer the wavelength) becomes. This is also kind of consistent with the picture below:



As we climb the well, the radial coordinates get ‘smaller’, and so the same lengthed ‘thing’ would be measured longer the further from the well we go. Of course radial coordinate and physical radial distance are different things, but whatever.

**Interior solution**

Now let’s solve for the metric in the interior of the star. We go back to Einstein’s equation, taking out the cosmological constant however:



And we will model the interior of the star as a perfect fluid so we have:



Now let’s work out the RHS of Einstein’s equation a bit,



Equating the LHS and RHS, first we see that since the LHS is diagonal, the RHS must be too – I guess assuming that **g** is diagonal, which seems a reasonable assumption here given isotropy. Therefore we must have that υi = 0. So let’s fill in the 0th component of the Einstein equation. Or not.