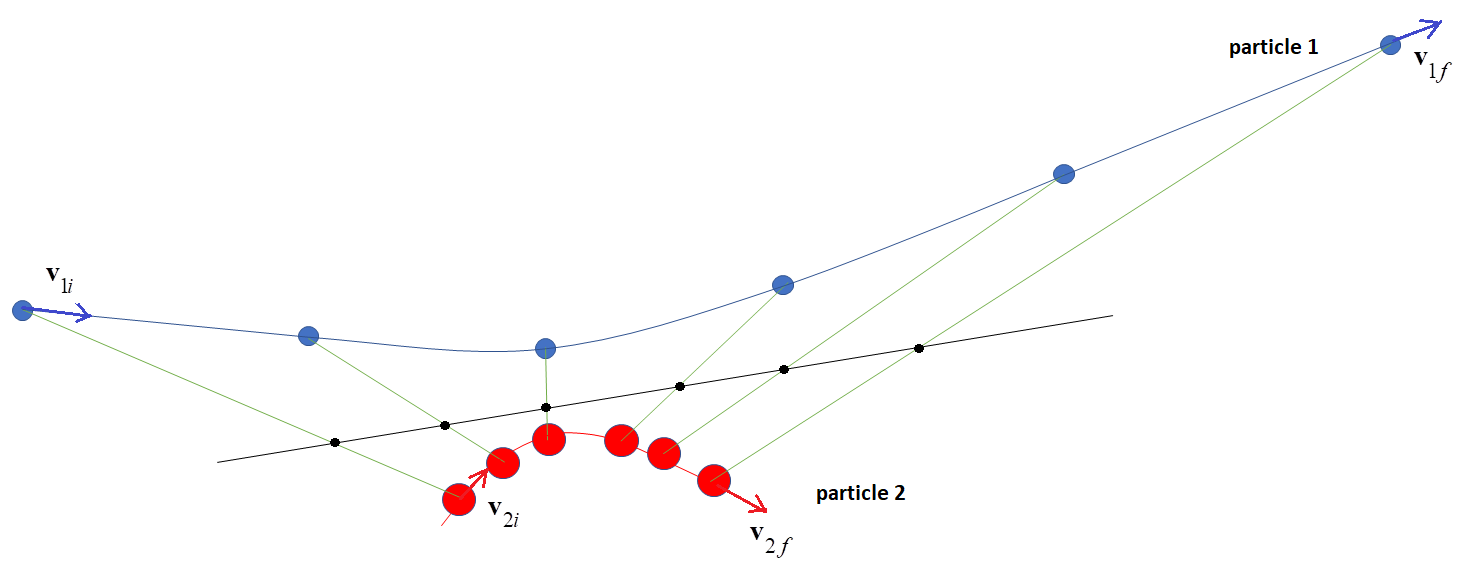
**2-Particle Interactions**

Now that we have all of these equations under our belt, let’s analyze collisions between particles. Generic collision looks like this I guess, at least for a repulsive interaction force. But we could’ve illustrated attractive forces, which might cause, say, the blue guy to swing around the red one instead.



The dot is the center of mass of the system, which we shall shortly ascertain must be traveling at a constant velocity. We’ll specialize to the case where there are no external forces acting on our particles. In that case, from N2L we have:



so that the total momentum of the system is conserved, and as such, the velocity of the center of mass is constant too. Also,



So the total angular momentum of the system is conserved. Additionally, from the work energy equation we have that:



Of course E is the *total* energy. The total energy can be written as the sum of mechanical energies and internal energy. And so we would have:



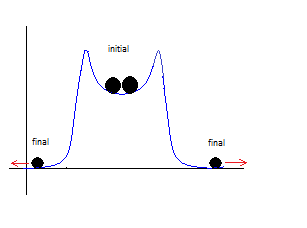
Could write this as:



And if we subsume ΔU into ΔEint, then we’d have, altogether:



ΔEint. can be positive or negative. It would be positive in most collisions, exemplified by the heating up of the two objects after the collision which means that some kinetic energy got converted to random kinetic (thermal) energy. If ΔEint. were negative then it would mean that some internal energy was lost and converted to mechanical kinetic energy - perhaps via a chemical explosive or something. Or even if there were no thermal energy to account for as internal, we could have ΔEint be positive or negative due to changes in potential energy U between the initial and final situations (due to changes in relative positions). In nuclear reactions we can have that be the case when two nuclear fragments are bound together in a kind of unstable potential well. Ultimately they will tunnel out of the well and lost internal (potential) energy will be converted to kinetic enerty.



But given **v**1i and **v**2ione still cannot in general solve these equations for **v**1f and **v**2f since these generally constitute more unknowns than we have equations. For instance in d dimensions these would be 2d unknowns, and only d+1 equations. So in 2D or 3D we’d be left with 1 or 2 unknowns still, which basically amounts to needing to know the orientations of the particles. Still, we can make some progress if we move to the center of mass reference frame. Before doing so, we’ll check and observe that moving to a different frame still preserve the equations. Let’s define the new frame velocity to be **v**frame. And let’s say = **v** – **v**frame. And so then the conservation equations read:



and,



So they both check out – the latter simply thanks to the conservation of momentum equation. The most advantageous frame to switch to is the c.o.m. frame because this annihilates the momentum term. The center of mass velocity is of course,



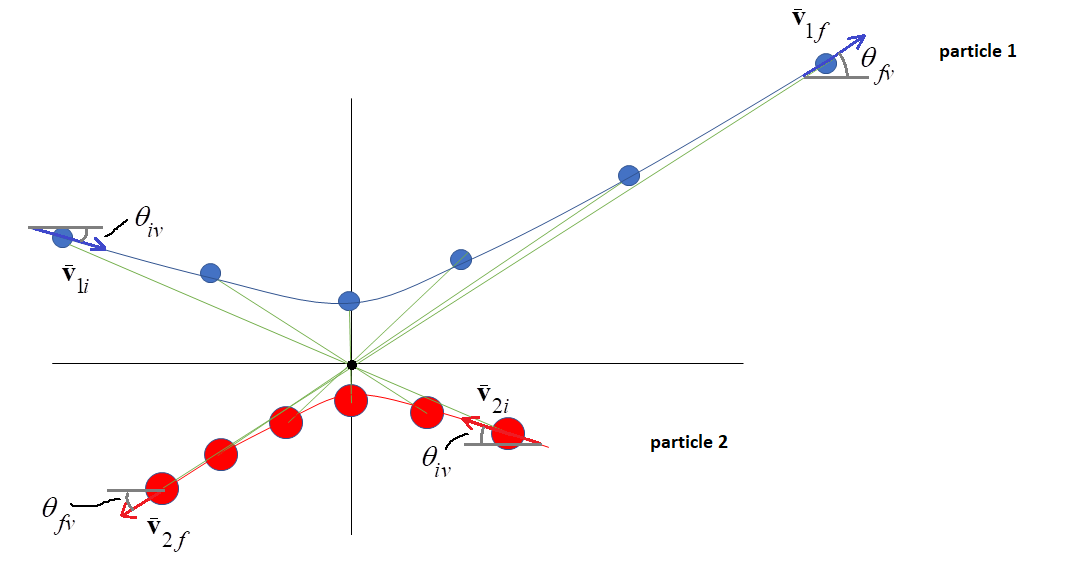
Then denoting the relative velocity by , we can say:



which simplifies to:



Since the relative momentum must add up to zero in both the initial and final situations, we can see that in this reference frame, the collision would look something like this below, where the velocities point in opposite directions both before, during, and after (note the velocities don’t have to lie along the same line though), since the velocities are negatives of each other, though with different magnitudes. Call the angle of 1i = - angle of 2i = θiv. And call the angle of 1f = - angle of 2f = θfv = θiv + θ. Then θ is called the scattering angle.



and the energy conservation equation would read,



But maybe it is worth simplifying even more:



Guess I could’ve anticipated this result. Anyway, so now we have for our two equations:



Plugging the momentum relation into this we have:



and then the other is given by (note minus sign present in the *vector* isn’t present in magnitude here, of course):



And now going back to the ‘lab’ frame, we have:



where KErel = (1/2)μ|**v**1i – **v**2i|2. I believe the coefficient √(1-ΔEint/KErel) is called the coefficient of restitution?

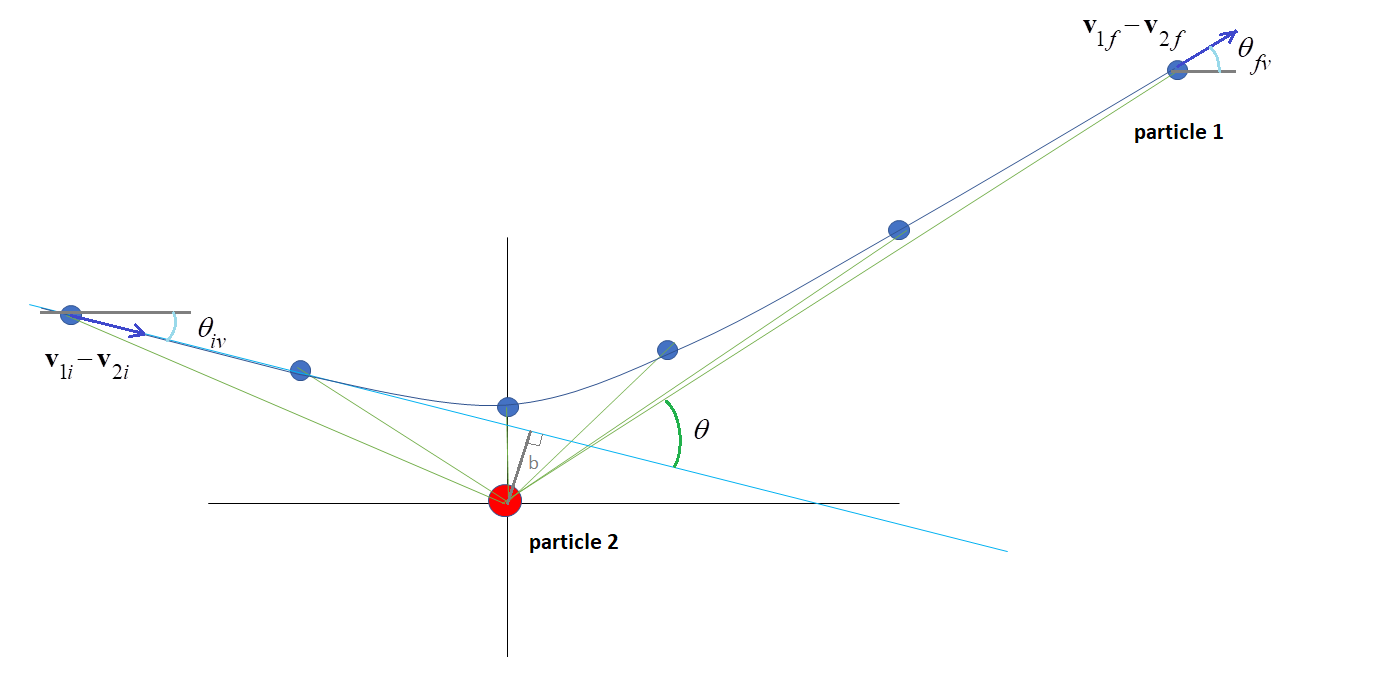


Worth pointing out that if ΔEint = 0, then δp = μ|**v**1i – **v**2i|. Of course we do not know which way δ**p** points, though we do know its magnitude. In 1D this isn’t a problem, because we *do* know – must be either 0 or 180°. In 2D or 3D we don’t, but we know that even in 3D, the trajectory will still be planar, as it is in 2D, due to angular momentum conservation (assuming central force). To borrow some work ahead, consider the relative angular momentum of the two particles:



And we know that since there is no relative torque, this relative angular momentum is conserved. But this means that magnitude and direction of **r** × **v** must always be same. In particular the direction must be the same at all times and so whatever plane is initially formed by these two before the collisions, will be the plane after the collision. So basically the two objects will never leave their initial plane of motion. So all 3D collisions are really 2D. And so δ**p** will only vary within that 2D plane. So there is really only one unknown for higher dimensional collisions (assuming we know ΔEint), and that’s the new angle within that plane. This can only be determined via knowledge of the particular force.

When we analyze collisions in more specific detail in the hard shell and inverse square interaction, we will be solving for the motion by changing coordinates to R = c.o.m. and r = r1 – r2. So we will be implicitly analyzing collisions from the perspective of stationary particle 2 (this is different from case below, because particle 2 will always be stationary, not just initially: our reference frame here is not inertial like it is below) The collision from this reference frame looks as below:



Anyway, this is of interest merely to note the angles which define **v**1i­ – **v­**2i and **v**1f – **v**2f­ are the same as those which define 1i and 1f, since the former vectors point in the same direction as the latter. So is the same in both the c.o.m. and relative reference frame. It’s more convenient to define it in terms of the relative reference frame variables. We can see from above that the final angle θfv is related to the initial via: θfv = θiv + θ where θ is the so-called scattering angle, and b the impact parameter – also the distance of closest approach. And so we can write:



So basically θ is the only thing for which we need specific knowledge of the force between the particles. And θ will depend on the impact parameter, b, among other things.

**Collision w/r to initial rest frame of either particle**

Going back to 2D collisions, let’s prove an interesting point – namely that a completely elastic collision between a moving particle and a stationary one will result in the two particles emerging along trajectories 90 degrees apart. This is most easily proven starting back at the beginning. Let’s take the conservation of momentum equation and dot both sides into themselves.



and now compare this with the energy equation.



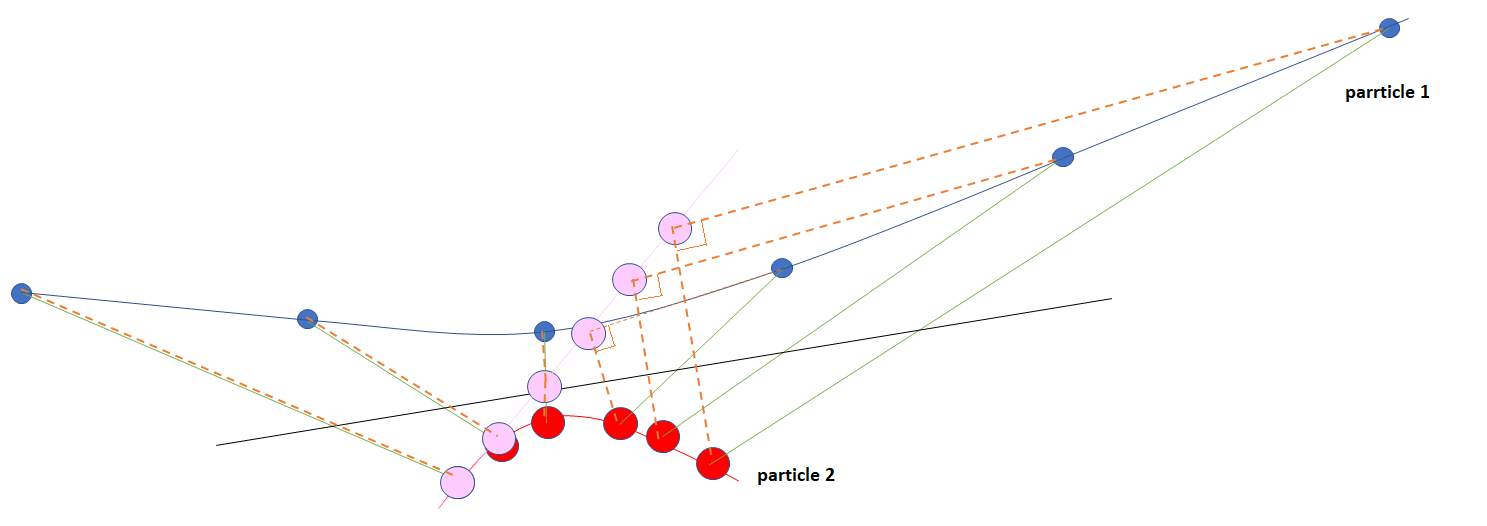
So we see that:



So, if the collision is elastic, and m2 is initially stationary so that **v**2i = 0, then we have:



meaning the particles emerge at a 90˚ angle with respect to each other. I guess you could generalize somewhat to say that *in the initial rest frame of either given particle*, they will split off at a 90 degree angle after the collision. So for instance, superimposing the initial rest frame of particle two (motion of that reference frame is the pink particle two) onto the collision at the top of the page, and drawing the positions with respect to that origin with dotted lines, we’d have something like:



and can see we do have roughly 90o separation of the products. We can show another interesting point, about how the initial velocity of the incoming particle relates to the maximum outgoing velocity of the initially stationary one. Let particle 1 be the incoming guy, and particle 2 be stationary. Then from the highlighted equations above, we have:



In the limit of large m1 >> m2, we have:



So the velocity of the incoming particle will be roughly unchanged, if it’s large compared to the target. And that the velocity of the target outgoing can range between 0 and 2**v**1i, depending on the angles. I guess 0 would be if the particle 1 didn’t interact with the target at all, and 2**v**1i would be for a head-on 1D collision. Glancing collisions will give something in between.

**Specialization to 1D**

So in 1D, the conservation equations alone are enough to solve for final velocities, assuming some ΔEint. Let’s take it to be zero, i.e., a fully elastic collision. And then say the first particle rebounds backwards upon striking the second particle. Then our equations come to:



and,



So yeah.

**Example**

Let’s now assume we don’t have zero ΔEint. Then,



And we’ll presume rebound too. So that makes δp negative, as whatever velocity it has after, it should be less then it was. So,



Filling that stuff in,



and,



and,



and I guess,



So there.

**Thermodynamics**

Another interesting consequence of the equations pertains to the approach to thermodynamic equilibrium. Let’s work out an expression for the difference in their final velocities:



If we were to assume that δ**p** averages to zero, then we could say that the difference in velocities would average to zero as well. And so on average, the particles will equilibriate to the center of mass velocities. Let’s work out the expression for the difference in final kinetic energies of two colliding particles:



if we say that m1 = m2, then this drastically simplifies. And likewise, if we were to say that δ**p** is isotropic again, then the kinetic energy difference would average to zero.