**Special Relativity Kinematics**

Now we’d like to introduce space-time position, velocity, momentum, etc., vectors, in preparation for writing down the dynamical equations in an invariant way. So first, the position vector.



This is given by:



It is invariant w/r to Lorentz transformation since:



And therefore its magnitude is invariant w/r to Lorentz transformations – as we know actually since this is:



which we previously observed was relativistically invariant. Now let’s define the space-time displacement vector:



And let’s calculate the magnitude of this vector, which is invariant as it must be:



If we specialize to the rest frame of the particle then we’ll have:



So we have the general result,



Before moving on, let’s note that for photons, Δs2 = 0 always, which implies that dτ = 0 so that there is no ‘proper’ time for a photon. Therefore this result only applies to particles moving at less than the speed of light, i.e. to time-like vectors. Now let’s define the space-time velocity. It is the rate of change of a particle’s space-time position w/r to time in its own rest frame time.



And this is given by:



The spatial part of the vector will reduce to the usual velocity vector at low speeds. Since this is a vector, it is invariant w/r to Lorentz transformations, and in particular its magnitude is invariant – as we discussed previously is true for any vector. Let’s work out its magnitude…we have (remember dot product of two time-like basis vectors is -1):



and so,



So we can see that the norm of the 4-velocity is –c2. Thus, all objects seem to travel through space-time at the same ‘rate’, which is the speed of light. This fact provides another point of view on the time-dilation. The rate at which one travels through space-time is constant, and thus as you speed up (increasing your rate of travel through space), your rate of travel through time must slow down (this is time-dilation) to keep your rate of travel through space-time constant. Finally here let’s observe that the space-time velocity of a photon would be ∞ since its rest time dτ = 0. Next let’s consider the space-time momentum. We can define the space-time momentum vector as:



This is given by:



(where we’ve peaked ahead and recognized the time-like component of is energy/c, and the spatial component is just the spatial momentum) For the magnitude (squared) of the vector we have:



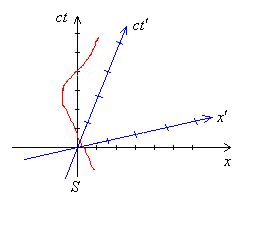
So we have:



If we tried to apply these formulas to a photon we would have two problems: one being that υ is ∞, and the other that m = 0. These two problems cancel each other out in a sense though because the space-time momentum vector is well defined. We just have to use the last definition of . And its magnitude squared is consequently 0.

**Example: velocity transformation laws**

The geometrical properties of these space-time vectors makes determining transformation laws easier in some respects. For instance, let’s determine the velocity addition law. So suppose we have an object thrown with speed **v** in reference frame S. What is its speed **v**′ in reference frame S′ moving with speed **u** (in the x-direction) w/r to S?



Well, since  is a space-time vector, its components will transform according to:



Anyway, so writing these components out we get:



which implies,

