**Scalars, Vectors and Tensors**

So we’re familiar with a vector. Generally we can write these as:



And we have a dot product between vectors two vectors, which gives a scalar (i.e. just a number). For instance,



and the cross product between two vectors gives another *vector*.



Well we can also define an *outer* product between two vectors. This gives us a *tensor*. The way it looks is this:



Note the outer product is not commutative, as,



and while the diagonal elements are the same, the off-diagonal ones aren’t necessarily. For instance the (AB)xy = AxBy and (BA)xy = BxAy, which aren’t necessarily equal. The lack of a commutative property is not too unusual, as the cross product isn’t commutative either.

Generally speaking, an arbitrary tensor **T** would have arbitrary components like this:



A tensor is said to be *symmetric* when its Txy = Tyx, Txz = Tzx, Tyz = Tzy. It is said to be anti-symmetric when Txy = -Tyx, Txz = -Tzx, Tyz = -Tzy. A special tensor is the unit tensor, which looks like this:



Now sometimes we want to take dot products between tensors and vectors. It would work like this. Instead of doing this out in full generality which would give me 27 terms, I’ll just do a special case – you’ll get the idea. So consider the tensor and vector:



Then



Or we could do:



which is different as you can see. The special thing about the unit tensor is that it always returns the vector it is dotted with so that **1**·**v** = **v**·**1** = **v**. For instance,



and likewise we would find that:



as well. We can also take cross products between tensors and vectors:



and we could work out **v**×**T** similarly. We can also take dot products between tensors themselves. For instance consider:



Then,



So you can see we get another tensor out of it. You can verify for yourself that the unit tensor keeps up the good work by satisfying **1**·**T** = **T**·**1** = **T**. Finally one often defines a double dot product which is written and worked out as below:



Of course it isn’t generally true that **T**1**:T**2 = 0 for arbitrary **T**1 and **T**2. Letting λ stand for any of the unit vectors, where λ = 1, 2, 3, we can write this contraction of a tensor as:



where δij is the Kronecker delta function, which equals 1 if i = j, and zero otherwise. Likewise we can see that:



And so we see that **T**1:**T**2 = **T**2:**T**1. Final note maybe. We will often deal with contractions like **T**:**AB**, where **T** is a tensor and **A**, **B** are vectors. We will often use the property that if **T** is a symmetric tensor, then this is equal to **T**:**BA** as well, since:



and therefore,



But if **T** is symmetric then this is:



So there we go. Pretty much all the tensors we’ll be working with are symmetric. By the way, we can represent this in matrix notation:



**Differential Operators**

Guess I’ll put this here. We often use vector shorthand to denote various differential operators,



and do things like,

