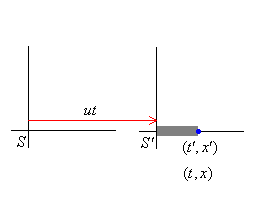
**Lorentz Transformation 1**

Given how the lengths and time contract, we can set up a coordinate transformation. First I’ll derive the actual mathematical form of the coordinate transformation, and then I’ll look at geometric representations of it.

**Mathematical representation of coordinate transformation**

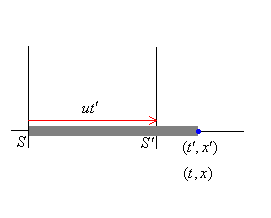
Let S be a stationary frame, and S′ be a frame moving to the right. And let their times coincide at t = t′ = 0, when the frames cross. Now then, if an event happens in frame S at the coordinates (*t,x*), where/when does the event happen in S′, i.e., what are the coordinates (*t*′,*x*′)?



We’ll use the following construction to infer the relationship between the positions/times. Consider a board (in grey) in the S′ reference frame with length x′. And suppose that a light is flashed at the end of the board at position (t′,x′) and (t,x) in S′ and S respectively. If you’re in S′, you’d measure the length of the segment between O′ and x′ to be x′ of course. If you’re in S, you’d measure the length of that segment to be x – ut, where x is the coordinate of that event, and ut is the distance to the origin of S′. So from the length contraction formula, we have x – ut = x΄/γ. Or maybe I can just say that you’d calculate the event to have occurred at position x = ut + x΄/γ, since you’ll measure the length of that rod to be x΄/γ and you’ll clock the length of time elapsed to be t. Either way, we can solve for x΄ and get:



So that’s how the x′ coordinate transforms. What about the t′ coordinate? Well now we must consider the situation from the opposite perspective. Let the distance x in the S reference frame be fixed. Then from the S′ perspective, this length will be contracted to x/γ since it is moving w/r to the S′ reference frame. Additionally, from the S′ perspective this distance would be ut′ + x′. And so we’d have x/γ = ut΄ + x΄.



Solving for x we get:



So now we want to solve for t΄ in terms of x and t. So we’ll solve for t΄ here and then substitute in what x is from the top equation:



So our coordinate transformations are:



which we can write more symmetrically as:



Let us define the generalized (4-position)



Then we can relate the coordinates of an event in frame S, to those in frame S′ using a little tensor notation. We use Λ to signify the Lorentz transformation since they both start with the same letter.



Note the inverse of this matrix is a transformation in the other direction (β → -β). For instance,



which is as we’d expect.

**Geometric representation of the coordinate transformation**

Let’s just focus on the t and x coordinates, since the others are the same. Then the coordinate transformation says,



Another way to write this is:



ξ is called the ‘rapidity’. So,



We’ll see this one again. But there’s another nice way to write it of more immediate interest. Let’s say φ = tan-1(β). Then,



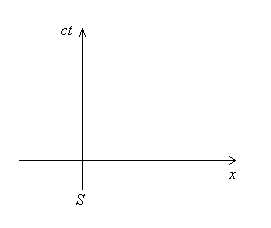
Finally we’ll define the prefactor δ = 1/√cos2φ as the ‘distortion’. It can be written as:



And so we can write the matrix as:



Continuing with this representation, let’s draw an S frame of reference.



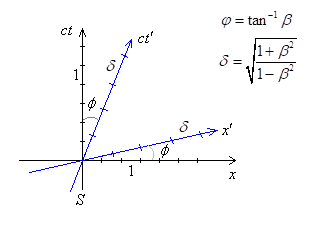
How does the S′ frame of reference, which is moving a speed u in the x-direction compare? Want to know where the x´ and ct´ axes lie within the ct-x plane. Well x´ axis is the locus of points in the ct-x plane for which ct´ = 0. This would imply,



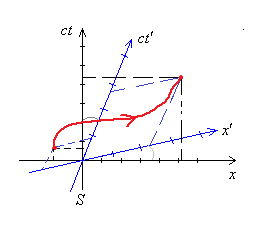
And ct´ axis is locus of points in the ct-x plane for which x´ = 0. This would imply,



Furthermore, the S´ axes are scaled differently than S. A unit of 1 on the S axis corresponds to a unit δ (the distortion factor) on the S´ axis. The distortion factor is 1 when v = 0, and goes to ∞ when v = c. And so this looks like,



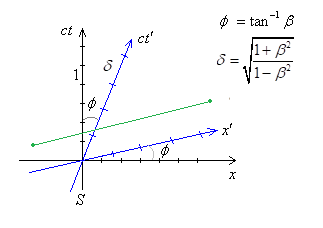
So the upshot is that relative to S’s coordinate axes, S′ ‘s coordinate axes are rotated inwards by an angle φ which approaches 45° as u → c, and also stretched by a factor of δ which approaches ∞ as u → c. The space-time coordinates would look like this, geometrically (never mind that our particle is sometimes going faster than c in this diagram)



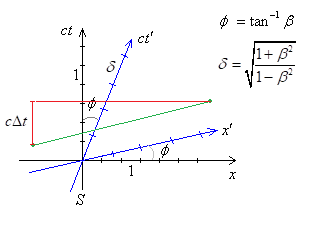
Observe that if S′ speed were to be c, then the x′ and ct′ axes would merge, erasing the distinction between space and time. Intresting.

**Simultaneity**

Events which are simultaneous in one frame are not in another. Consider the moving frame. The green bar represents events that are simultaneous according to it.



But we can see that the left event (left green dot) will have occurred before the right event (right green dot) in the S frame of reference. Further, we can see that the temporal separation between the two events would be:



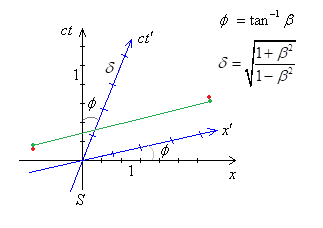
as follows (remember δ is the distortion, a unitless factor)



and so,



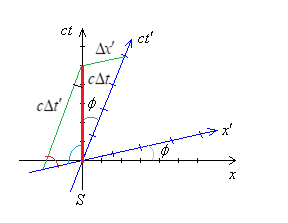
This matches what we found in an earlier file, though in that file, it was S with respect to which the events were simultaneous, not S´ and so prime and unprime were reversed. One nice thing to note, however, is that causality is always preserved, so if an event occurs in a certain sequence in one frame, it will occur in the same sequence in any other. Consider the two red dots,



In the S´ reference frame, the left occurs before the right. This is also true in the S reference frame.

**Time-dilation reprised**

Now let’s look at a variety of predictions of SR from a more geometrical frame of reference. First we’ll consider time-dilation. Consider a space-time diagram for two different reference frames, S and S′, with S′ moving with speed u to the right w/r to S. *Hard way*…And consider two events that occur at the same point in space in S. Then the time interval between these events is called the proper time, Δt. The S space-time displacement vector between these two events is illustrated in red. Let’s also look at the time interval in S´. The S´ space-time displacement vector components between these two events is illustrated in green.



and let’s see what time Δt´ it corresponds to in the S reference frame. First note that black angle is φ. Blue angle is 180 – φ – (90 - 2φ) = 90 + φ. Red angle is 180 – φ – (90 + φ) = 90 - 2φ. Using the law of sines we have:



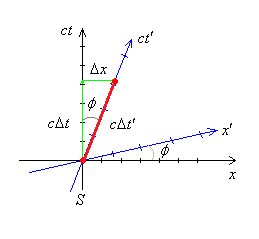
Now we note that:



and so we have:



which is of course the time dilation formula. *We can do this the other (better) way too*. Let the event be proper in the S´ frame rather, and break its components down into the S frame.



We have:



And also have:



Filling this in,



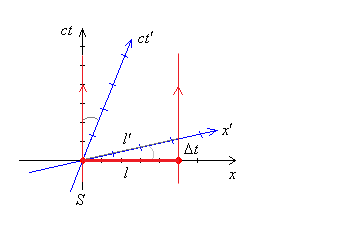
which is the correct result relating the proper time to the improper time, though the notation is opposite to what we used prior. Or even easier…



So there we go.

**Length contraction reprised**

Now let’s look at length contraction. Suppose we have some length in the S reference frame, the rest frame. The length of the object in S´ is *not* the x component of the vector, unlike how it Δt´ was simply the t component of the vector. That’s because length measurements must be made *at* *the same time*. In order for S´ to measure the length between the two events S´ must perceive the two ends simultaneously. To that end, we can plot the world lines of each end of the rod, and see where they intersect the S´ frame. And that length will be the perceived length of the rod.



So the space-time geometry says that:



and also we have:



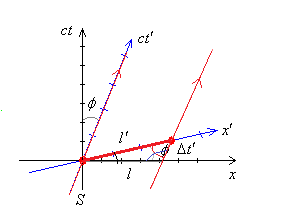
Filling this in we have:



and so we see that lengths in the moving frame are contracted by a factor of γ. Or even more succintly, we can just say:



*Again, we can do this the other way (harder way now)*. Let the length now be proper in the S´ frame.



We consider where the ends world lines intersect the S x-axis. Like before we have the red angle is 90 - 2φ, blue angle is 90 + φ, and black angle is φ. From the geometry (law of sines) we have:



Again,



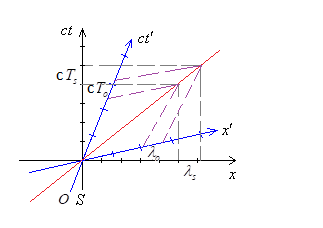
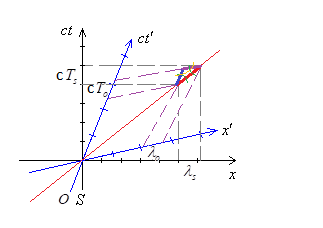
So,



which is the length contraction formula, though with notation reversed from prior.

**Doppler shift reprised**

Now let’s take a look at this one. Say our frame S´ is moving with speed u to the right. And we have a light wave with frequency f moving towards S´. We’ll say that it goes through the origin at time t = t´ = 0. So what is the frequency f´ as measured in S´?

Need to get relationship between To and Ts. Consider the highlighted triangle.



Law of cosines says,



Now



So,



Inverting, we have:

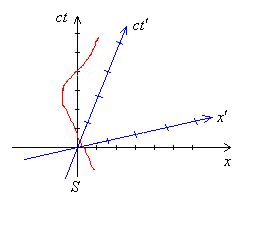


Here we have to recognize that we’re headed towards the wave, evidently, since it appears in positive coordites. And so this accounts for the – sgn discrepancy between this expression and our standard one from before. Accordingly, setting β → -β, we find:



**Velocity transformation**

The geometric approach isn’t always the easiest way to do things however. Consider this situation. A particle travels with speed v according to reference frame S which is relatively at rest. What then is the speed of the particle w/r to S′ moving with speed u w/r to reference frame S?



We can answer this by simply forming the velocity in each of the respective reference frames. For instance in S′, the position of the particle is some x′(t′) and so its velocity is:



Filling in the coordinates of S′ in terms of S we have:



So we have:



We’ll note that this law shows explicitly that in no reference frame can c be exceeded. For instance, suppose we shine a light to the right in reference frame S. Then the speed of light in reference frame S′ moving at speed c to the left is:



Another observation we can make is that this law reduces to the Galilean transformation law when speed u and v are much less than light in which case we get approximately v′ = v-u.