**Special Relativity Dynamics**

Now we’ll look at a relativistically correct formulation of dynamics.

**Newton’s 2nd law modified to accommodate special relativity**

First we’ll consider Newton’s second law. As originall stated, we have:



This seems to indicate that constant application of a force can accelerate an object up to any velocity. But we know now that c is ultimate speed limit. Therefore these equations cannot be correct, or rather it cannot be the case that p = mv. We *can* keep this form actually, we just have to use the correct ‘time’. We should use the proper time of the object, τ, rather than the time of the rest frame. So then we’d have:



Changing the definition of momentum then, we can rewrite this as the new relativistically correct version of N2L.



We’ll note that as v → 0, γ → 1 and so this definition reduces to the classical one at relativistically small speeds. Let’s play around with this formula a bit,



which gives us a relatively nice form, properly interpreted.



and so we can explicitly see that as the speed of the object decreases to 0, and consequently γ → 1, this equation reduces to the usual version of N2L.

**Special case of circular motion**

Observe that the correction to the expected result is of smaller order than the first term. But in general, the force will be directed along **a** and **v**. Two special cases. If **v** is perpendicular to **a**, like in circular motion, then we get,



**Special case of linear motion**

If **v** is parallel to **a**, like with straight line motion, then we get,



**Example: magnetic field**

We can apply the second expression to the case of charges rotating in a magnetic field,



and so,



as before, but with p redefined.

**Example: constant force**

Consider the motion of a particle under a constant force, in 1D, starting from rest. Then we have,



These equations can be integrated.



Solving for v,



which can also be integrated to get position. We get,



**WE equation modified to accommodate special relativity**

We also need to modify the work-energy equation. According to Physics 1,



But given a large enough amount of work, this would again imply that we can move the particle up to a velocity greater than c. So what is the correct expression for the energy of a moving particle? This can be deduced from the definition of **p**. Consider,



and so we can say that, separating the work done into work done by conservative and non-conservative forces:



This is our new relativistically correct work-energy equation. It is convenient to note that we can also write the energy as:



which you may verify. Okay well let’s do it,



Let’s examine what E is when v = 0. In that case we have,



This is called the rest energy is the energy. On the other hand, if a particle has no mass, but momentum (how something could have momentum but no mass might be a mystery, but photons have this property), then we’d have,



Incidentally, we can define the KE as the difference between E and E0 – the energy purely due to motion. And so we’d have,



**Example**

Suppose we accelerate a proton through a potential difference of 10MV. What will be its velocity?



and so we get,



Note that for an electron,



and proton,



So this would translate to:



**Relativistically invariant formulation of N2L + WE**

Now we’d like to write these equations out in a relativistically invariant way, meaning we want to write these equations in terms of space-time vectors. So consider:



We can write these two equations as one space-time vector equation as follows:



So then defining the space-time force as:



we can combine the work-energy equation and N2L into a single relativistically invariant equation:



where τ is the time measured in the particles rest frame as always. Be careful to note that this expression is equivalent to the former two. There is nothing wrong with those two and we can use them or this one at our leisure. This is just another way of writing it.

**Least action modified to accommodate special relativity**

As has been found in the past, having a least action formulation to the relativistic laws of physics does prove useful. So what Lagrangian would give the correct relativistically invariant equations of motion? We can verify that the following choice does the job.



which is as before, except that the KE has been replaced by a negative factor of the total particle energy. For instance, let’s write down the equations of motion, neglecting the EM field,



which is simply:



as required.

**Example**

Consider the motion of a particle under a constant force, in 1D, starting from rest. Then we have,



formulating the equation of motion we have,



Solving for v gets us,



Integrating again gives us:



Expanding for a small force we get,



which is as we’d expect. The Lagrangian formulation often makes the analysis a little easier though.

**Relativistically invariant least action principle**

The Lagrangian above is perfectly adequate, but a little asymmetric in the sense that it treats time and position in different ways. It would be nice if we could write a least action principle that treated time and space symmetrically, like the relativistically invariant formulation of N2L and such that we accomplished above. What would such a Lagrangian look like? Well it should be relativistically invariant scalar. And it should have something to do with the energy obviously. A scalar quantity that is related to the energy is magnitude of the energy-momentum space time vector, or relatedly the magnitude of the space-time velocity vector (could use either one in this case). So if we were to try to guess a form for the relativistically invariant Lagrangian we might formulate the following guess, neglecting external forces:



Let’s apply the Euler-Lagrange equations of motion to this entity,



And we see that this does indeed work, i.e. it gives the same equations as



in a force free environment.