**2 Particle Interactions (Orbits)**

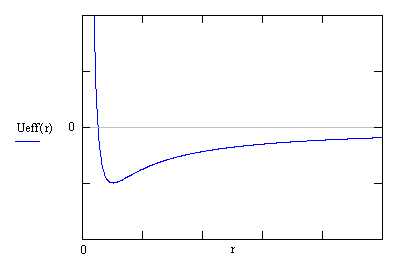
Let’s do a qualitative examination of orbits. We’ll go back to:



It is useful to combine angular kinetic energy and gravitational potential energy into one term and define Ueff.(r) = ℓ2/2μr2 + U(r), and to define the radial kinetic energy KEr = μvr2/2. Then we can write:



Now let’s display a typical plot of Ueff.(r), using U(r) = -k/r, though we could easily do this for any other potential.



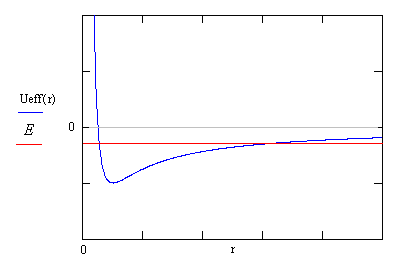
Generally, it goes to ∞ as r → 0, and then to 0 as r → ∞, as you can tell from the formula itself. And we can see there is a local minimum at:



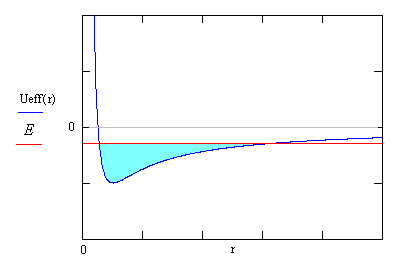
and the value of that energy is:



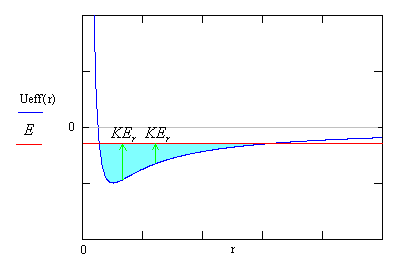
Now the energy, , is a constant which must always be greater than or equal to Ueff(r), since vr2 must always be positive or 0. Let’s display a typical plot for then,



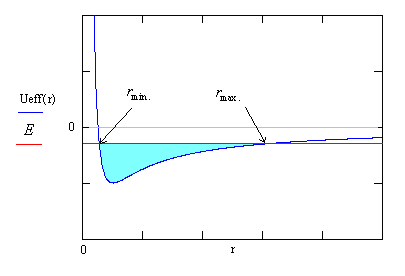
Now since the energy of the particle μ can only be greater than or equal to Ueff.(r), the particle can only exist in the region where the red line is above the blue curve, and this is highlighted below:



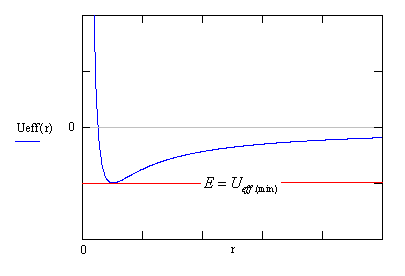
Outside this region is called the forbidden zone. The difference between  and U­eff.(r) is the radial kinetic energy. Two points in the path, and their respective radial KE values are shown below:



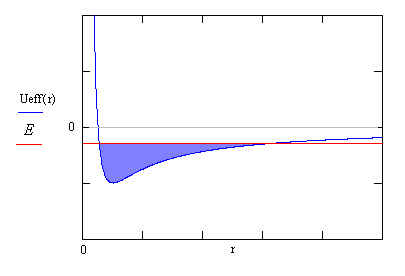
We will note that there are two special points located at the intersection of  and Ueff.(r). These are where KEr = 0, and hence are the so-called turning points, where the particle starts to turn back around and go in the other direction. These points would be the distance of furthest and closest approach.



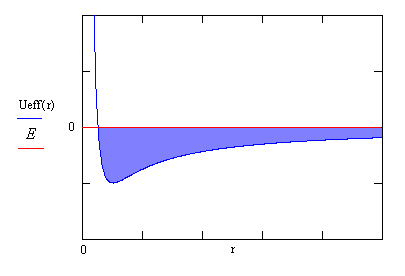
Judging from the graph, there are 4 special cases. Case 1 is when  = minimum of Ueff., shown below.



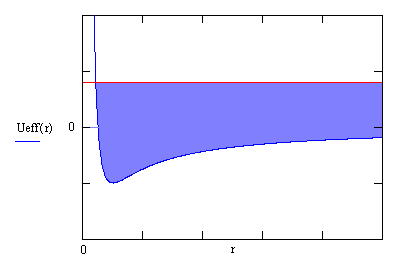
In that case there is only one allowed radius, r = r­min.. And we can see that the orbit must be circular for this energy, since a circle is the only curve with just one radial value. Case 2 is where Ueff.(min) < < 0. In this case there are two allowed radii. We will see later that this orbit corresponds to an ellipse.



Case 3 is when = 0, shown below. Such orbits we’ll find to be parabolic orbits. These orbits are not closed, and so the particle μ would make come by for a distance of closest approach but then ultimately fly off into space, never to come back.



Similarly, we have case 4 which is when > 0. These orbits we’ll find to be hyperbolic. Again there would be a distance of closest approach, but ultimately they would fly away and never come back.



**Circular orbits**

Suppose we locate an asteroid (m = 1.47×1015 kg) in the vicinity of the Sun (M = 2×1030 kg) a distance of r0 = 3×1011m away from the Sun’s center. And suppose its velocity is measured to be: v0 = 2.11×104m/s t. Then what kind of orbit will it have, and what will be its distance of closest approach, furthest approach to the Sun?

Well first, we’ll calculate, and ℓ



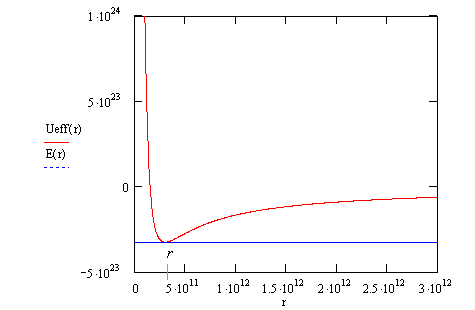
and ℓ is:



Now we’ll plot  and Ueff.(r), which is:



Obtaining,



**Elliptical orbits**

Suppose we give the asteroid velocity, v0 = 2.11×104 t + 1.5×104 r. Then what kind of orbit will it have, and what will be its distance of closest approach, furthest approach to the Sun?

To visualize the answer, let’s go back and recalculate , and ℓ.



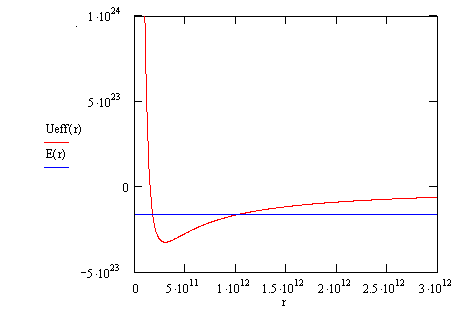
and ℓ is unchanged since we didn’t change vt or r0.



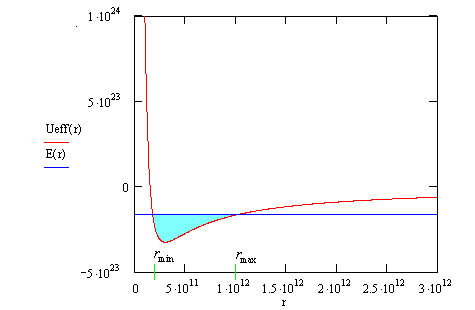
Now let’s plot, and Ueff.(r) which is also unchanged



as a function of position.



Now since must be greater than or equal to Ueff.(r), the only radii which the asteroid can have are the ones for which > Ueff.(r), i.e., the blue shaded region.



So this indicates that the asteroid will orbit the Sun with radii between rmin and rmax. Therefore, it will be orbiting elliptically. These radii can be determined numerically from a graphing calculator. They can also be solved for analytically. Observe that the min/max radii occur when KEr = 0. This means that,



and so,



reworking this a little we have:



**Parabolic/hyperbolic orbits**

Now let’s take the same asteroid and boost its radial velocity even more, so that. v0 = 2.11×104 t + 2.5×104 r. Now what will its orbit look like?

Again, let’s calculate and ℓ.



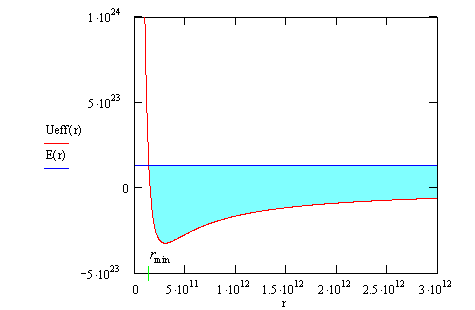
(note is positive) and ℓ is unchanged since we didn’t change vt or r0.



Now let’s plot and Ueff.(r) which is also unchanged



as a function of position, as well as r­­min and the permitted region.



Since > 0, there is no finite rmax. Therefore the orbit is unbounded. It is hyperbolic. The distance of closest approach can be obtained as above using the + sign (the – sign just gives a negative r value which doesn’t make any sense in the context).



We will get parabolic orbits when the energy is 0 and hyperbolic orbits for anything higher. There are some observations we should make. First, if is negative, then the orbit is closed. It will be either circular or elliptical. A circular orbit has the minimum possible energy. Elliptical orbits have more energy than circular ones. As the energy of the orbit increases, the eccentricity of the ellipse will increase more and more; that is rmin and rmax will become more and more separated. If becomes positive, the asteroid will have enough energy to break away from the Sun. Then the orbit is open, and parabolic.

**Example**

An Ariane 6 rocket launches into space from French Guiana with a payload of multiple small satellites. At an altitude of 800 km it injects the first satellite into orbit. The velocity of the satellite at that instant is 10 km/s, and the velocity makes an angle of 82o with the geocentric radius vector. Given that RE = 6.37×103 km and ME = 5.97 × 1024 kg, calculate the minimum height of the satellite above the earth's surface during its orbit.

So we’ll go back to:



We’re looking for the radial extrema, so set dr/dt = 0,



and solve for r,



Let’s just take μ = m,



The minimum radius is therefore,



Can write this as:



Now let’s get the initial energy, angular momentum, etc.,



Filling all of these in,



**Stable orbits**

Now we’d like to investigate how stable the closed orbits above are. We’d like to know, which central forces can support stable circular orbits? Basically, we need to know which attractive potentials will be such that the effective potential has a local minimum. So let our potential be:



Then the effective potential is:



It will have a local extremum where,



It will be stable when the 2nd derivative there is positive, i.e., when,



So we must have n < 2 in order to support a stable circular orbit.