**Navier – Stokes equation**

Now let's fill our equation for **π** into N2L and continuity equation to come to the Navier Stokes equation. Let’s start with the N2L equation, and then plug in the stress/strain equation (using Einstein Summation Notation).



Now the continuity equation reads,



Applying this equation eliminates the ρ derivatives in N2L, giving us:



So we have,



This equation would be useful if we already know v and want to get P, or if we already know P and want to get v. Otherwise, some extra thermodynamics will have to go into getting P, as is done for the sound wave example down below. The operator on the right is often defined as the so-called 'convective' derivative.



Further, if we assume our liquid is incompressible, then the divergence of **v** must be 0, in

which case we get the so-called Euler equation.



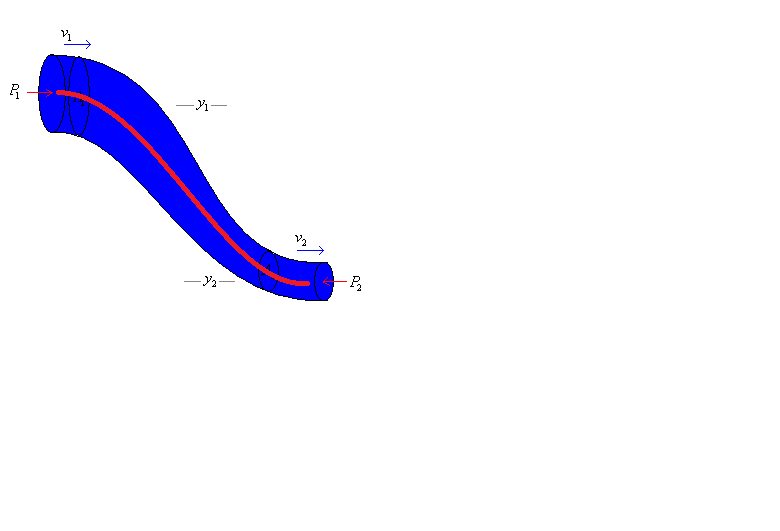
And if we assume no viscosity, then we get:



Before we get carried away with applying the full differential equation to the motion of fluids, let's recognize that often times, in simple cases, use of one of the three equations (continuity, N2L, WQE) alone often suffices to determine a solution. So sometimes we will use them, and sometimes the Navier-Stokes equation itself (which of course was derived from them and is thus neither more general nor less). First let's consider the familiar Bernoulli.

**Bernoulli’s equation**

Derive Bernoulli’s equation for a non-viscous incompressible fluid.



So consider a tube of flow, where we have a pressure P1 at the top and P2 at the bottom. The different elevations are marked, and the velocities of the water at the top and bottom are given. Bernoulli’s equation basically relates P1, y1, v1 to P2, y2, v2. To find the relationship, integrate the NS equation along the center of the tube. We are integrating because the NS equation is a differential equation which tells us precisely how the velocity is changing with position along the tube. But all we want is information about the tube's endpoints. So to get information about just the endpoints we integrate from one end to the other, just like ∫f'(x) = f(b) - f(a) gives information about the endpoints. Then we have:



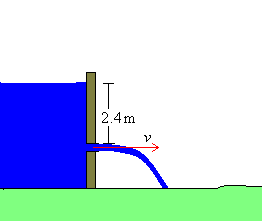
which can be rearranged as:



where we define the kinetic energy *density* and potential energy *density* of a fluid, κε = 1/2ρv2, and pε = ρgy respectively. Let’s make a few observations. First, when the velocity of the fluid increases, Δκε increases, and therefore ΔP decreases – which is to say it goes down. So when a fluid moves faster its pressure decreases. Also, when Δpε increases, that is, when its height increases, the pressure tends to drop as well. This last statement we’re familiar with from our study of *static* fluids.

**Example**

To generate hydroelectricity, one may construct a damn, and then construct a pipe that allows water to spill out of the damn, the kinetic energy of the falling water is then converted to electric potential energy (electricity) in much the same way as the angular kinetic energy of a windmill is converted to electricity (you’ll perhaps learn about these methods in physics 2). Suppose that that our pipe is located 2.4m below the damn. How fast does the water pour out of the damn. If the diameter of the pipe is 50cm, what power is delivered by the pipe, i.e., how much KE is flowing out of the pipe per second?



We will use Bernoulli’s equation. Let us consider point 1 to be at the top of the damn, and point 2 to be at the opening on the side. The pressure at points 1 and 2 is Patm. because at each point the water is exposed to the atmosphere alone. Further, the water at the top of the damn is moving very slowly, so its velocity is approximately 0. Proceeding…



Observe that this is the same velocity as would be had if the water had fallen directly through a vertical distance of d. The kinetic energy flowing out of the pipe, per unit time is:



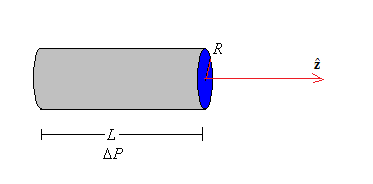
Now the flow rate is f.r. = Av = π(0.25)2(6.9) = 1.36 m3/s. Therefore the power is:



If the machinery transferring the kinetic energy to electric potential energy were perfectly efficient, then this damn would generate 32.4 kW of electrical power. This is a pretty small damn however. Hoover damn for instance is 750ft high, and generates around 2GW power.

**Poiseuille’s Law**

Now let’s consider Poiseuille’s law, which relates the pressure drop across a viscous fluid to its flow velocity. So assume we have a pipe with radius R, and length L.



We only have a z-velocity, and it varies in the r direction only (and not with time). Plus, the pressure only varies in the z direction as well. So the equation for vz comes to, in cylindrical coordinates:



Now integrate this equation from z = 0 to z = L (since we want Poiseuille’s law)



Now let’s solve for vz,



c must be zero because vz would blow up at the origin otherwise. So then we have:



Now we should have that vz(R) = 0 so…



Plugging this in we have:



And now we want to relate the average velocity to the pressure difference. So averaging…



So we have:



Well, actually the law relates the flow rate to the pressure difference. But I leave it at this. Let's ask one more question. Suppose we wanted to know the total force exerted by the pipe on the fluid, at the surface, between z = 0 and z = L. Well the pressure at any point would be:



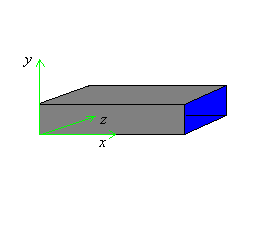
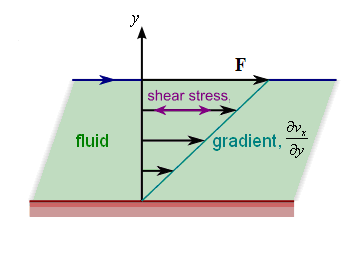
And so the net force on the fluid over the entire length is:



And this is to be expected because the net force on the cross-section ends of the pipe is the same, which is how we get this steady state flow.

**Viscous fluid flow though wide rectangular pipe**

Now let’s consider fluid flow along a wide rectangular pipe, that looks something like that on the right.

We’d like to know how the velocity of the fluid depends on its vertical position from the center of the fluid. We’ll use the incompressible fluid Navier-Stokes equation. We’ll also note that we have no net external forces.



Where we note that only vx is non-zero. ρ and vx do not depend on time or x. vx does depend on y though, and P depends on x (this is needed to sustain flow in the x-direction). Now since Pth.(x) and vx(y) and these to functions of different variables are equal to each other, we must have that they are both equal to a constant. So we must have:



Assuming the flow is 0 at the boundaries, then:



So then,



**Bernoulli-Poiseuille Law ??**

I’d like to try to combine the two. Let’s start with:



This isn’t working as well. How about going to the Poiseuille analysis, but allowing vz to change:



Now integrate this equation from z = 0 to z = L (since we want Poiseuille’s law)



Now let’s solve for vz,



c must be zero because vz would blow up at the origin otherwise. Then we’d try to integrate, but we can’t really because we don’t know how vz2 depends on r.

**Sound waves**

Now let's try to analyze the propagation of sound through some fluidic medium. We would like to determine an equation for the speed of sound. We start with the NS equation:



and we will treat our fluid as a non-viscous, and mostly incompressible so that we can ignore the (η + λ) term as usual. Of course sound waves rely on the fluid compressing, but it turns out that the Pth. term is much bigger than the (η+λ) term so that it may be safely neglected in comparison. We may also ignore body forces as gravity plays little role here. Also, we will assume the velocity varies only in the x-direction. So then this simplifies to:



Now if we knew Pth.. and ρ, then we could get v and be done. But we do not know these since they are varying in the sound wave as well. So we need two more equations. One equation we can go back to is the continuity equation:



Assuming flow only in the x-direction, we have:



and another, which relates pressure to ρ and v is the bulk-modulus equation:



where B is the bulk modulus of the fluid/liquid. Dividing top and bottom of the fraction by N and taking the difference to be differentials we may write:



And now we're set. However, solving for Pth., ρ, and v would be generally very difficult and result in a non-linear partial differential equation. So what we would like to do instead is to determine a perturbative solution to our equations. You will see this approach used in quantum mechanics when you take it. So the idea is to write our solution as Pth. = P0 + P1, v = v0 + v1, and ρ = ρ0 + ρ1, and assume that these corrections P1, v1 and ρ1 are *small*. That is why its a 'perturbative' solution. So let's plug these into our 3 equations and see what we get. Starting with NS:



where we keep terms only to first order. Now doing same with continuity equation:



and our bulk-modulus equation,



Now we're ready to solve our 3 equations. Its just like solving 3 equations and three unknowns. In the first equation there is P1. Let's attempt to eliminate this variable. So take our B equation and divide both sides by dx,



and plug into the NS equation



Now let's try to eliminate the ρ1 term. There is one equation we haven't yet used - the continuity equation. So let's apply d/dx to both sides of continuity equation,



and apply d/dt to both sides of new NS equation:



Now multiply both sides of our new continuity equation by -B/ρ0 and add to new NS equation,



and so this is our equation for v1.



This is the familiar wave equation. And so it follows directly that the velocity of the sound wave is:



One can look up the bulk modulus in tables if you want to know particular values. This formula applies well to gasses, liquids, and even solids. Let's specialize a little to an ideal gas, kind of like air. What is B for it? We assume that the compression of the air particles in the medium happens quickly so that no heat is transferred to or from the air while this happens. Certainly the air doesn't heat up appreciably as a sound wave goes by. So that means that during this process the air will follow the adiabatic equation of state: PbVbγ = PaVaγ. Let point 'a' be the point before the compression happens, in which case Pa = P0. Point 'b' will be at some particular point in the processes; then we have Pb = P0 + P1. Similarly Va = V0, and Vb = V0 + V1. Filling these in, and solving for P1 to first order we get:



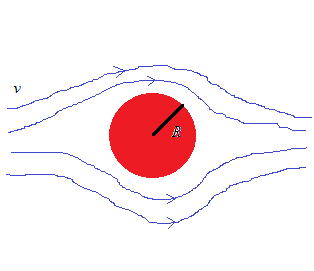
and so we can identify B = γP0 for air. Air is roughly a diatomic molecule, for which γ = 1.4. So this would make the speed of sound of air at sea level to be:



which is about as close as one can get to the true value.

**Flow around a sphere moving through fluid with velocity v.**

We will analyze this problem by analyzing its converse. We will switch perspectives and consider the sphere stationary, and the fluid flowing past at velocity v.  I call this velocity v∞ to distinguish it from the fluid velocity field v, and also to remind that the velocity field will go to v∞ asymptotically.



Our equation is...



We will assume no external forces - gravity is unimportant here. And we will as usual assume no expansion. Let's look to the steady state solution so that **v** doesn't depend on time. Then our equation is:



We also presume we have ‘creeping’ flow, so the term second order in v is ignored (maybe keeping this term would generate the turbulent drag force?).



This, in conjunction with our incompressibility presumption,



Should allow us to work out the velocity field **v**(r). But I don’t want to. Suffice to say, that after a bit of work, we will find, in spherical coordinates and spherical vector components,



where R is the radius of our sphere. In purely vector format this comes to:



We can fill this back into our top NS equation to get the pressure,



Then we would calculate the stress tensor **π**,



And integrate it around our sphere to get the net force. We find:



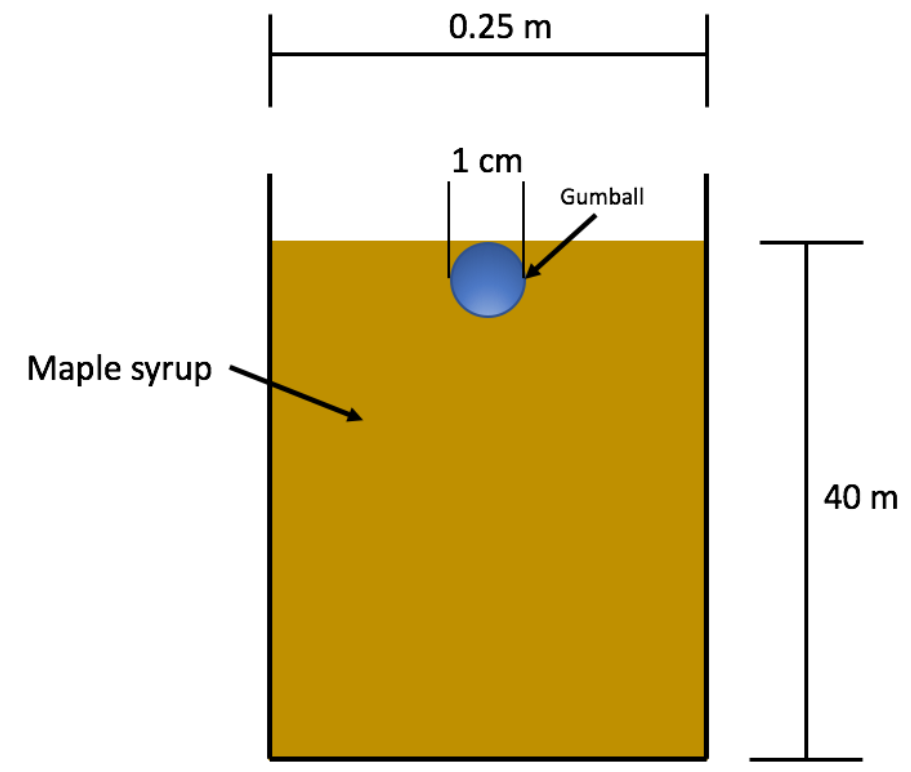
Of course v∞ is just the velocity of our object. So we’ll say:



Note this works for laminar flow, but not so much for turbulent flow. In the latter case, the force goes as v2.

**Example**

A gumball with a mass of 2g is released from rest into the situation shown in the image. At what time t will the gumball reach a velocity of v=0.50m/s?



The equation for our gumball would be, accounting for gravity, drag, and the buoyant force:



(negative sign on 6πηRv is because v is negative) We’ll try a solution of the form v(t) = Ae-αt + B,



So,



And since it starts from rest, that fixes A:



Solving for time,



So there. Filling in values we get t = 0.06s I think. If we use η = 0.2 Pa∙s, and ρms = 1370kg/m3.

**Example**

Show that for a bucket of water revolving at rate Ω, the surface of the water takes the shape of a paraboloid. So in a rotating reference frame, we can update the Navier-Stokes equations by adding in fictitious forces. Recall we found that for particles,



So we’d change:



to (updating m -> ρ):



(we presume the origin of our coordinate system is not moving, and also that our reference frame is rotating at a constant speed, not accelerating) Assuming steady-state, incompressibility, and that, well, v = 0 in our reference frame, and that our actual force is just gravity, we have:



Let’s fill in Ω = Ω, and use cylindrical polar coordinates.



Solving for the pressure, we have:



I think we can say that at (r,z) = (0, 0), the pressure is atmospheric, so:



Now the pressure at the top of the surface must be atmospheric pressure. Let z(r) be the formula for the position of the surface. Then we need:



So that’s our equation for the surface, which is indeed a paraboloid.